J. Math. Anal. Appl. 373 (2011) 83–93

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Li–Yorke and distributionally chaotic operators

T. Bermúdez [a], A. Bonilla [a], F. Martínez-Giménez [b], A. Peris [b,*]

[a] Departamento de Análisis Matemático, Universidad de La Laguna, 38271, La Laguna (Tenerife), Spain
[b] IUMPA, Universitat Politècnica de València, Departament de Matemàtica Aplicada, Edifici 7A, 46022 València, Spain

ARTICLE INFO

Article history:
Received 18 May 2010
Available online 12 June 2010
Submitted by Richard M. Aron

Dedicated to Professor Antonio Martinón on his 60th birthday

Keywords:
Li–Yorke chaos
Distributional chaos
Irregular vector
Distributionally irregular vectors
Weighted shift operators

ABSTRACT

We study Li–Yorke chaos and distributional chaos for operators on Banach spaces. More precisely, we characterize Li–Yorke chaos in terms of the existence of irregular vectors. Sufficient "computable" criteria for distributional and Li–Yorke chaos are given, together with the existence of dense scrambled sets under some additional conditions. We also obtain certain spectral properties. Finally, we show that every infinite dimensional separable Banach space admits a distributionally chaotic operator which is also hypercyclic.

1. Introduction

During the last years many researchers paid attention to the "wild behavior" of orbits governed by linear operators on infinite dimensional spaces (more especially, on Banach or Fréchet spaces). One of the most significant cases being the *hypercyclicity*, that is, the existence of vectors $x \in X$ such that the orbit $\operatorname{Orb}(T, x) := \{x, Tx, T^2x, \ldots\}$ under a (continuous and linear) operator $T : X \to X$ on a topological vector space (usually, Banach or Fréchet space) X, is dense in X. We refer the reader to the recent books [2] and [16] for a thorough account of the subject. This notion from operator theory joined chaos after the definition of Devaney [11], which (in our context) requires hypercyclicity and density of the set of periodic points of T in X.

The concept of "chaos" appeared for the first time in the mathematical literature in the paper of Li and Yorke [20] of mid 70's.

Definition 1. Let (X, d) be a metric space. A continuous map $f : X \to X$ is called *Li–Yorke chaotic* if there exists an uncountable subset $\Gamma \subset X$ such that for every pair $x, y \in \Gamma$ of distinct points we have

$$\liminf_n d\big(f^n x, f^n y\big) = 0 \quad \text{and} \quad \limsup_n d\big(f^n x, f^n y\big) > 0.$$

In this case, Γ is a *scrambled* set and $\{x, y\} \subset \Gamma$ a *Li–Yorke* pair.

* Corresponding author.
E-mail addresses: tbermude@ull.es (T. Bermúdez), abonilla@ull.es (A. Bonilla), fmartinez@mat.upv.es (F. Martínez-Giménez), aperis@mat.upv.es (A. Peris).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.06.011

Li–Yorke chaos was studied for linear operators, for instance, in [12,14]. Incidentally, every hypercyclic operator $T : X \to X$ on a Fréchet space X is Li–Yorke chaotic with respect to any (continuous) translation invariant metric d on X: E.g., fix a hypercyclic vector $x \in X$ and consider the segment $\Gamma := \{\lambda x;\ |\lambda| \leqslant 1\}$, which is a scrambled set for T.

The notion of distributional chaos was introduced by Schweizer and Smital in [28]. The definition was stated for interval maps with the intention to unify different notions of chaos for continuous maps on intervals. This concept was widely studied by several authors and can be formulated in any metric space. In particular, it is considered for linear operators defined on Banach or Fréchet spaces in [25,22,17–19].

For any pair $\{x, y\} \subset X$ and for each $n \in \mathbb{N}$, the *distributional function* $F^n_{xy} : \mathbb{R}^+ \to [0, 1]$ is defined by

$$F^n_{xy}(\tau) = \frac{1}{n}\,\mathrm{card}\bigl\{0 \leqslant i \leqslant n-1:\ d\bigl(f^i x, f^i y\bigr) < \tau\bigr\}$$

where card$\{A\}$ denotes the cardinality of the set A. Define

$$F_{xy}(\tau) = \liminf_{n\to\infty} F^n_{xy}(\tau),$$
$$F^*_{xy}(\tau) = \limsup_{n\to\infty} F^n_{xy}(\tau).$$

Definition 2. (See [28,26].) A continuous map $f : X \to X$ on a metric space X is *distributionally chaotic* if there exist an uncountable subset $\Gamma \subset X$ and $\varepsilon > 0$ such that for every $\tau > 0$ and each pair of distinct points $x, y \in \Gamma$, we have that $F^*_{xy}(\tau) = 1$ and $F_{xy}(\varepsilon) = 0$. The set Γ is a *distributionally ε-scrambled* set and the pair $\{x, y\}$ a *distributionally chaotic pair*. Moreover, f exhibits *dense distributional chaos* if the set Γ may be chosen to be dense.

Generally speaking, for any distinct $x, y \in \Gamma$ the iterations of these points are arbitrary close and ε separated alternatively, but additionally there are time intervals where any of these excluding possibilities is much more frequent than the other.

Given $A \subset \mathbb{N}$, its *upper and lower densities* are defined by

$$\overline{\mathrm{dens}}(A) = \limsup_n \frac{\mathrm{card}\{A \cap [1, n]\}}{n}, \quad \text{and} \quad \underline{\mathrm{dens}}(A) = \liminf_n \frac{\mathrm{card}\{A \cap [1, n]\}}{n},$$

respectively. With these concepts in mind, one can equivalently say that f is distributionally chaotic on Γ if there exists $\varepsilon > 0$ such that for any $x, y \in \Gamma$, $x \neq y$, we have

$$\underline{\mathrm{dens}}\bigl\{n \in \mathbb{N};\ d\bigl(f^n x, f^n y\bigr) < \varepsilon\bigr\} = 0, \quad \text{and} \quad \overline{\mathrm{dens}}\bigl\{n \in \mathbb{N};\ d\bigl(f^n x, f^n y\bigr) < \tau\bigr\} = 1,$$

for every $\tau > 0$.

From now on X will be a Banach space and $T : X \to X$ a bounded operator. In this case the associated distance is $d(x, y) = \|x - y\|$, with $x, y \in X$, where $\|\cdot\|$ is the norm of X. We recall the following concept from operator theory.

Definition 3. (See Beauzamy [3].) A vector $x \in X$ is said to be *irregular* for T if $\liminf_n \|T^n x\| = 0$ and $\limsup_n \|T^n x\| = \infty$.

Inspired by this definition, and by the notion of distributional chaos, we consider the following stronger property.

Definition 4. A vector $x \in X$ is said to be *distributionally irregular* for T if there are increasing sequences of integers $A = (n_k)_k$ and $B = (m_k)_k$ such that $\overline{\mathrm{dens}}(A) = \overline{\mathrm{dens}}(B) = 1$, $\lim_k \|T^{n_k} x\| = 0$ and $\lim_k \|T^{m_k} x\| = \infty$.

2. Li–Yorke chaotic operators

We first discuss Li–Yorke chaos for operators with the following result that establishes the equivalence between Li–Yorke chaos and the existence of an irregular vector.

Theorem 5. *Let $T : X \to X$ be an operator. The following assertions are equivalent*:

(i) *T is Li–Yorke chaotic.*
(ii) *T admits a Li–Yorke pair.*
(iii) *T admits an irregular vector.*

Proof. That (i) implies (ii) is clear; in order to prove that (ii) implies (iii) suppose that T admits $y, z \in X$ with

$$\liminf_n d\bigl(T^n y, T^n z\bigr) = 0 \quad \text{and} \quad \limsup_n d\bigl(T^n y, T^n z\bigr) > 0.$$

If we set $x = y - z$ then $\liminf_n \|T^n x\| = 0$ and there is $\delta > 0$ such that $\limsup_n \|T^n x\| > \delta$. If $\limsup_n \|T^n x\| = \infty$ then x is an irregular vector. Otherwise, $M := \limsup_n \|T^n x\| < \infty$.

We observe that $\|T\| > 1$ since, given $n, m \in \mathbb{N}$, $n < m$, with $\|T^n x\| < \delta/2$ and $\|T^m x\| > \delta$, we have that $\|T\|^{m-n} > 2$. We select a strictly increasing sequence of integers $(n_k)_k$ such that the sequence of vectors $(T^{n_k}x)_k$ tends to 0 fast enough so that

$$\left\|T^{n_{2k}}x\right\| < 4^{-k},$$

$$\sum_{j=0}^{k} \frac{\|T^{n_{2j}+n_{2k+1}}x\|}{4^j \|T\|^{n_{2j-1}} \|T^{n_{2j}}x\|} < \left\|T^{n_{2k}}x\right\|, \quad \text{and} \tag{1}$$

$$\frac{\delta}{4^k \|T\|^{n_{2k-1}} \|T^{n_{2k}}x\|} - \sum_{j=0}^{k-1} \frac{M}{4^j \|T\|^{n_{2j-1}} \|T^{n_{2j}}x\|} > k, \tag{2}$$

for all $k \in \mathbb{N}$, where $n_{-1} = n_0 := 0$. We now set

$$u = \sum_{j=0}^{\infty} \frac{1}{4^j \|T\|^{n_{2j-1}} \|T^{n_{2j}}x\|} T^{n_{2j}}x.$$

We will see that u is an irregular vector for T. Indeed,

$$\begin{aligned}
\left\|T^{n_{2k+1}}u\right\| &\leqslant \sum_{j=0}^{\infty} \frac{\|T^{n_{2j}+n_{2k+1}}x\|}{4^j \|T\|^{n_{2j-1}} \|T^{n_{2j}}x\|} \\
&< \left\|T^{n_{2k}}x\right\| + \sum_{j=k+1}^{\infty} \frac{\|T^{n_{2k+1}}\| \|T^{n_{2j}}x\|}{4^j \|T\|^{n_{2j-1}} \|T^{n_{2j}}x\|} \\
&< 4^{-k} + \sum_{j=k+1}^{\infty} \frac{\|T\|^{n_{2k+1}}}{4^j \|T\|^{n_{2j-1}}} < 2^{-k},
\end{aligned}$$

by (1) and the selection of $T^{n_{2k}}x$, $k \in \mathbb{N}$. On the other hand, let $(m_k)_k$ be a strictly increasing sequence of integers such that $\|T^{m_k}x\| > \delta$, $k \in \mathbb{N}$. Without loss of generality we suppose that $n_1 < m_1 < n_2 < m_2 < \cdots$ with $m_{2k} - n_{2k}$ tending to infinity. By (2) we obtain that

$$\begin{aligned}
\left\|T^{m_{2k}-n_{2k}}u\right\| &\geqslant \frac{\|T^{m_{2k}}x\|}{4^k \|T\|^{n_{2k-1}} \|T^{n_{2k}}x\|} - \sum_{j \neq k} \frac{\|T^{m_{2k}-n_{2k}+n_{2j}}x\|}{4^j \|T\|^{n_{2j-1}} \|T^{n_{2j}}x\|} \\
&> \frac{\delta}{4^k \|T\|^{n_{2k-1}} \|T^{n_{2k}}x\|} - \sum_{j=0}^{k-1} \frac{M}{4^j \|T\|^{n_{2j-1}} \|T^{n_{2j}}x\|} - \sum_{j=k+1}^{\infty} \frac{\|T\|^{m_{2k}-n_{2k}}}{4^j \|T\|^{n_{2j-1}}} \\
&> k - \sum_{j=k+1}^{\infty} \frac{1}{4^j} > k - 4^{-k},
\end{aligned}$$

for all $k \in \mathbb{N}$. Thus $\limsup_n \|T^n u\| = \infty$ and u is an irregular vector.

Finally, to see (iii) implies (i), if $u \in X$ is an irregular vector for T, then $S := \operatorname{span}\{u\}$ is a scrambled set for T by a direct application of the definitions. □

Prăjitură studied in [27] some properties of operators having irregular vectors. By [27] and Theorem 5, we obtain certain properties of Li–Yorke chaotic operators.

Corollary 6. *Let $T : X \to X$ be a Li–Yorke chaotic operator. The following assertions hold*:

1. $\sigma(T) \cap \partial\mathbb{D} \neq \emptyset$.
2. *T^n is Li–Yorke chaotic for every $n \in \mathbb{N}$.*
3. *T is not compact.*
4. *T is not normal.*

Definition 7. An operator $T : X \to X$ satisfies the *Li–Yorke Chaos Criterion* (LYCC) if there exist an increasing sequence of integers $(n_k)_k$ and a subset $X_0 \subset X$ such that

(a) $\lim_{k\to\infty} T^{n_k}x = 0$, $x \in X_0$,
(b) $\sup_n \|T^n|_Y\| = \infty$, where $Y := \overline{\operatorname{span}(X_0)}$ and $T^n|_Y$ denotes the restriction operator of T^n to Y.

One might expect that the LYCC is a sufficient condition for Li–Yorke chaos. It turns out that it is in fact a characterization of this phenomenon.

Theorem 8. *Let $T : X \to X$ be an operator. The following are equivalent*:

(i) *T is Li–Yorke chaotic.*
(ii) *T satisfies the Li–Yorke Chaos Criterion.*

Proof. Suppose T is Li–Yorke chaotic, by Theorem 5 we find an irregular vector $x \in X$. By setting $X_0 = \{x\}$, we verify the LYCC easily.

Conversely, suppose T satisfies the LYCC, let $(n_k)_k$, and $X_0 \subset X$ be the respective sequence of integers and subset of vectors satisfying conditions (a) and (b) of Definition 7. If there is $x \in X_0$ such that x is an irregular vector, then we are done. Otherwise we observe that every $u \in \operatorname{span}(X_0)$ satisfies $\lim_k T^{n_k}u = 0$ and $\sup_n \|T^n u\| < \infty$. Passing to subsequences of $(n_k)_k$ and $(m_k)_k$, if necessary, by property (b) and since $\sup_n \|T^n|_Y\| = \infty$, we can obtain a sequence $(u_j)_j$ of normalized vectors in $\operatorname{span}(X_0)$ such that

(a)′ $\|T^{n_k}u_j\| < \frac{1}{j}$, $j = 1, \ldots, k$, $k \in \mathbb{N}$, and
(b)′ $\|T^{m_j}u_j\| > 3^j M_{j-1}$, $j > 1$,

where $M_k := \sup\{\|T^n u_i\|;\ i = 1, \ldots, k,\ n \geqslant 0\} < \infty$, $k \in \mathbb{N}$. Without loss of generality, we may suppose that $m_1 < n_1 < m_2 < n_2 < \cdots$. We select any infinite subset $I \subset \mathbb{N}$ such that, for each $j \in \mathbb{N}$, if $i \in I$ with $i > j$, then $2^i > 2^j\|T\|^{n_j}$. The vector

$$u := \sum_{i\in I} \frac{1}{2^i} u_i$$

is well defined since the series is convergent. We will see that u is an irregular vector for T. On the one hand,

$$\begin{aligned}
\|T^{m_j}u\| &\geqslant \frac{1}{2^j}\|T^{m_j}u_j\| - \sum_{i\in I,\, i\neq j} \frac{1}{2^i}\|T^{m_j}u_i\| \\
&> \frac{3^j M_{j-1}}{2^j} - \sum_{i\in I,\, i<j} \frac{M_{j-1}}{2^i} - \sum_{i\in I,\, i>j} \frac{1}{2^i}\|T^{m_j}u_i\| \\
&> \left(\frac{3^j}{2^j} - 1\right) M_{j-1} - \sum_{i\geqslant j} \frac{1}{2^i} \xrightarrow[j\to\infty,\, j\in I]{} +\infty.
\end{aligned}$$

On the other hand,

$$\|T^{n_j}u\| \leqslant \sum_{i\in I,\, i\leqslant j} \frac{j^{-1}}{2^i} + \sum_{i\in I,\, i>j} \frac{1}{2^i}\|T^{n_j}u_i\| < \frac{1}{j} + \frac{1}{2^{j-1}}, \quad j \in I,$$

which shows that u is an irregular vector. □

3. The strong criterion for distributional chaos and spectral properties

The following criterion for distributional chaos was introduced in [17]. Since, for several reasons that will be clarified soon, this criterion is somehow very restrictive, we will call it the "strong" criterion. In this section we will study the spectral properties of operators that satisfy this criterion.

Definition 9. An operator $T : X \to X$ satisfies the *Strong Distributional Chaos Criterion* (SDCC), if there is a constant $r > 1$ such that for any $m \in \mathbb{N}$, there exists $x_m \in X \setminus \{0\}$ satisfying

(i) $\lim_{k\to\infty} \|T^k x_m\| = 0$,
(ii) $\|T^i x_m\| \geqslant r^i \|x_m\|$ for $i = 1, 2, \ldots, m$.

Such r is said to be a *SDCC-constant* for the operator T.

Theorem 10. *(See [17, Theorem 3.3].) Let $T : X \to X$ be an operator. If T satisfies the Strong Distributional Chaos Criterion, then T is distributionally chaotic.*

By $\mathbb{D}$ we mean the open unit disc, its boundary is $\partial\mathbb{D}$ and the complement of the closed unit disc will be written as $\overline{\mathbb{D}}^c$. We denote by $r(T)$ the spectral radius of T.

Proposition 11. *Let $T : X \to X$ be an operator. The following properties hold:*

(a) *If there exists $r > 1$ such that for all $m \in \mathbb{N}$ there exists $x_m \in X \setminus \{0\}$ with*

$$\left\| T^i x_m \right\| \geqslant r^i \|x_m\|, \quad \text{for all } i = 1, \ldots, m, \tag{3}$$

then $r(T) \geqslant r$.

(b) *(See [13, Lemma 6.4(b)].) If $\sigma(T) \subset \overline{\mathbb{D}}^c$ and $\lim_{k\to\infty} \|T^k x\| = 0$, then $x = 0$.*

Proof. (a) Assume that $r(T) < r$. Let $\varepsilon > 0$ be such that $r(T) + \varepsilon < r$. Then there exists $m \in \mathbb{N}$ such that for all $n \geqslant m$ we have that $\|T^n\|^{1/n} \leqslant r(T) + \varepsilon < r$. Moreover, by (3) for $m + 1$ there exists $x \in X \setminus \{0\}$ such that $\|T^i x\| \geqslant r^i \|x\|$ for $i = 1, \ldots, m + 1$. Then $\|T^{m+1} \frac{x}{\|x\|}\| \geqslant r^{m+1}$, so $\|T^{m+1}\|^{1/m+1} \geqslant r$, which is a contradiction. □

Corollary 12. *If $T : X \to X$ satisfies the Strong Distributional Chaos Criterion with SDCC-constant r, then $r(T) \geqslant r$.*

Theorem 13. *If T satisfies the Strong Distributional Chaos Criterion with SDCC-constant r, then the following properties hold:*

(a) *For any r_0 with $1 \leqslant r_0 \leqslant r$ we have $\sigma(T) \cap \partial D(0, r_0) \neq \emptyset$.*
(b) *There are not disjoint closed subsets F_1 and F_2 of $\mathbb{C}$ such that $F_1 \subset D(0, r)$ and $F_2 \subset \overline{\mathbb{D}}^c$ with $F_1 \cup F_2 = \sigma(T)$.*

Proof. (a) Assume that there exists $r_0 \leqslant r$ such that $\sigma(T) \cap \partial D(0, r_0) = \emptyset$. Take the spectral decomposition $\sigma_1 = D(0, r_0) \cap \sigma(T)$ and $\sigma_2 = \overline{D(0, r_0)}^c \cap \sigma(T)$. Let X_i and T_i with $i = 1, 2$ be given by the above spectral decomposition such that $X = X_1 \oplus X_2$, $T = T_1 \oplus T_2$ with $\sigma_i = \sigma(T_i)$, $i = 1, 2$. Let $m \in \mathbb{N}$. Then there exists $x_m = x_m^1 \oplus x_m^2 \in X \setminus \{0\}$ such that $\|T^k x_m\| = \|T_1^k x_m^1\| + \|T_2^k x_m^2\| \to 0$ as k tends to infinity and $\|T^i x_m\| \geqslant r^i \|x_m\|$ for $i = 1, \ldots, m$. Using the same argument as in [13, Lemma 6.4(b)] we have that $\|x_m^2\| \leqslant \|T_2^k x_m^2\|$, which tends to 0 as $k \to \infty$, that is, $x_m^2 = 0$. Hence $x_m = x_m^1$ and $\|T^k x_m\| = \|T_1^k x_m^1\| \geqslant r^i \|x_m\|$ for $i = 1, \ldots, m$. Henceforth, $r(T_1) \geqslant r$ but this is a contradiction since $\sigma(T_1) \subset D(0, r_0)$.

(b) Let $F_1 \subset D(0, r)$ and $F_2 \subset \overline{\mathbb{D}}^c$ be closed subsets of $\sigma(T)$ such that $F_1 \cap F_2 = \emptyset$ and $F_1 \cup F_2 = \sigma(T)$. Let X_i and T_i with $i = 1, 2$ be given by this spectral decomposition, i.e., $X = X_1 \oplus X_2$, $T = T_1 \oplus T_2$ with $F_i = \sigma(T_i)$, $i = 1, 2$. Let $m \in \mathbb{N}$. Then there exists $x_m = x_m^1 \oplus x_m^2 \in X \setminus \{0\}$ such that $\|T^k x_m\| = \|T_1^k x_m^1\| + \|T_2^k x_m^2\| \to 0$ as k tends to infinity and $\|T^i x_m\| \geqslant r^i \|x_m\|$ for $i = 1, \ldots, m$. By part (b) of Proposition 11 we have that $x_m^2 = 0$. Then by part (a) of Proposition 11 we obtain that $r(T_1) \geqslant r$ which is a contradiction since $\sigma(T_1) = \sigma_1 \subset D(0, r)$. □

An operator $S : X \to X$ is called *strictly singular* if for every infinite dimensional subspace M of X, the restriction of S to M is not a homeomorphism. An example of strictly singular operator is the compact operators.

Notice that a small compact perturbation of the unit operator could be distributionally chaotic [18, Proposition 3.5]. Also, in [18, Proposition 3.3] is proven that any compact perturbation of scalar operator cannot satisfy the SDCC. In the next result we improve that property, that is, if S is a strictly singular operator, then $S + \lambda I$ does not satisfy the SDCC.

Corollary 14. *If $S : X \to X$ is a strictly singular operator and $\lambda \in \mathbb{C}$, then $\lambda I + S$ does not satisfy the SDCC.*

Proof. It is well know that the spectrum of strictly singular operator is at most a countable sequence of eigenvalues that converges to zero as its only limit point. Hence the result is a consequence of part (b) of Theorem 13. □

A *hereditarily indecomposable* Banach space is an infinite dimensional space such that no subspace can be written as a topological sum of two infinite dimensional subspaces. W.T. Gowers and B. Maurey constructed the first example of a hereditarily indecomposable space [15]. In particular, they proved that every operator T defined on a hereditarily indecomposable space X can be written as $T = \lambda I + S$, where $\lambda \in \mathbb{C}$ and S is a strictly singular operator.

The class of hereditarily indecomposable spaces appeared for the first time in relation with hypercyclicity and chaos in [9], where it was proved that some complex separable Banach spaces admit no Devaney chaotic operator.

Corollary 15. *There are no operators satisfying the SDCC on any hereditarily indecomposable Banach space.*

4. Distributionally chaotic operators

Our purpose in this section is to study distributional chaos in connection with the existence of distributionally irregular vectors, and to give useful criteria for distributional chaos. First of all, we obtain the analogous of one of the implications in Theorem 5.

Proposition 16. *If $T : X \to X$ admits a distributionally irregular vector $x \in X$, then T is distributionally chaotic.*

Proof. If $x \in X$ is a distributionally irregular vector for T, then there exist increasing sequences of integers $A = (n_k)_k$ and $B = (m_k)_k$ such that $\overline{\mathrm{dens}}(A) = \overline{\mathrm{dens}}(B) = 1$, $\lim_k \|T^{n_k}x\| = 0$ and $\lim_k \|T^{m_k}x\| = \infty$. Let $S := \mathrm{span}\{x\}$. If $y, z \in S$, $y \neq z$, then $y - z = \alpha x$ with $\alpha \neq 0$. Therefore,

$$\left\|T^{n_k}y - T^{n_k}z\right\| = |\alpha|\left\|T^{n_k}x\right\| \xrightarrow{k\to\infty} 0, \quad \text{and}$$
$$\left\|T^{m_k}y - T^{m_k}z\right\| = |\alpha|\left\|T^{m_k}x\right\| \xrightarrow{k\to\infty} \infty,$$

which yields that S is a distributionally ε-scrambled set for T, for every $\varepsilon > 0$. $\square$

Problem 1. Does every distributionally chaotic operator $T : X \to X$ admit a distributionally irregular vector?

The following result from [17] yields a sufficient condition for distributional chaos, less restrictive than the SDCC.

Theorem 17. *(See [17].) Let $T : X \to X$ be an operator. If for any sequence of positive numbers $(C_m)_m$ increasing to ∞, there exists $(x_m)_m$ in $X \setminus \{0\}$ satisfying*

(a) $\lim_{n\to\infty} T^n x_m = 0$, *and*
(b) *there is a sequence of positive integers $(N_m)_m$ increasing to ∞, such that*

$$\lim_{m\to\infty} \frac{1}{N_m} \mathrm{card}\left\{0 \leqslant i < N_m;\ \left\|T^i x_m\right\| \geqslant C_m\|x_m\|\right\} = 1,$$

then T is distributionally chaotic.

We introduce a variation of the above criterion by allowing to have different sequences of vectors in part (a) and (b) of the above theorem.

Definition 18. An operator $T : X \to X$ satisfies the *Distributional Chaos Criterion* (DCC) if there exist sequences $(x_m)_m$ and $(y_m)_m$ in $X \setminus \{0\}$ with $y_m \in \overline{\mathrm{span}\{x_k;\ k \in \mathbb{N}\}}$ satisfying

(a) $\lim_{n\to\infty} T^n x = 0$, $x \in X_0$, and
(b) there is a sequence of positive integers $(N_m)_m$ increasing to ∞, such that

$$\lim_{m\to\infty} \frac{1}{N_m} \mathrm{card}\left\{0 \leqslant i < N_m;\ \left\|T^i y_m\right\| \geqslant m\|y_m\|\right\} = 1.$$

Observe that our criterion has, a priori, weaker requirements than the criterion of Cao, Cui and Hou in Theorem 17. We will show that they are actually equivalent.

Theorem 19. *Let $T : X \to X$ be an operator. The following properties are equivalent*:

(i) *T satisfies the hypothesis of Theorem* 17.
(ii) *T satisfies the Distributional Chaos Criterion.*

Proof. We only need to prove that (ii) implies (i). By (ii) there exists $(x_m)_m \subset \mathrm{span}(X_0) \setminus \{0\}$ that satisfies the condition (a) of Theorem 17 by linearity, and condition (b) by density. $\square$

Under the assumptions of the DCC we can ensure the existence of distributionally irregular vectors.

Proposition 20. *If $T : X \to X$ satisfies the DCC, then T admits a distributionally irregular vector.*

Proof. By Theorem 19, passing to subsequences if necessary, we find a sequence $(x_m)_m$ of normalized vectors in X and an increasing sequence $(N_m)_m$ of integers with $N_m - N_{m-1}$ tending to infinity such that

$$\frac{1}{N_m}\operatorname{card}\left\{0 \leqslant i < N_m\colon \left\|T^i x_m\right\| > m\|T\|^{N_{m-1}}\right\} > 1 - \frac{1}{m}, \tag{4}$$

$$\frac{1}{N_m}\operatorname{card}\left\{0 \leqslant i < N_m\colon \left\|T^i x_k\right\| < \frac{1}{m}\right\} > 1 - \frac{1}{m^2}, \quad k = 1, \dots, m-1, \tag{5}$$

where $N_0 := 1$.

Since our hypothesis obviously implies that $\|T\| > 1$, the series

$$x := \sum_k \frac{1}{\|T\|^{N_{2k-1}}} x_{2k}$$

is convergent, thus $x \in X$. We will show that x is a distributionally irregular vector for T. Indeed,

$$\begin{aligned}\left\|T^i x\right\| &\geqslant \frac{1}{\|T\|^{N_{2m-1}}}\left\|T^i x_{2m}\right\| - \sum_{k\neq m} \frac{1}{\|T\|^{N_{2k-1}}}\left\|T^i x_{2k}\right\| \\ &> 2m - \frac{1}{2m}\sum_{k<m}\frac{1}{\|T\|^{N_{2k-1}}} - \sum_{k>m}\frac{1}{\|T\|^{N_{2k-1}-N_{2m}}} \xrightarrow[m\to\infty]{} \infty\end{aligned}$$

if $i < N_{2m}$, $\|T^i x_{2m}\| > 2m\|T\|^{N_{2m-1}}$, and $\|T^i x_{2k}\| < \frac{1}{2m}$, $k < m$. Conditions (4) and (5) above imply

$$\frac{\operatorname{card}\{0 \leqslant i < N_{2m};\ \|T^i x_{2m}\| > 2m\|T\|^{N_{2m-1}},\ \|T^i x_{2k}\| < \frac{1}{2m},\ k < m\}}{N_{2m}} > \frac{m-1}{m},$$

and we obtain an increasing sequence of integers $B = (m_k)_k$ such that $\overline{\operatorname{dens}}(B) = 1$ and $\lim_{k\to\infty}\|T^{m_k}x\| = \infty$. On the other hand,

$$\left\|T^i x\right\| \leqslant \sum_k \frac{1}{\|T\|^{N_{2k-1}}}\left\|T^i x_{2k}\right\| < \frac{1}{2m+1}\sum_{k\leqslant m}\frac{1}{\|T\|^{N_{2k-1}}} + \sum_{k>m}\frac{1}{\|T\|^{N_{2k-1}-N_{2m}}} \xrightarrow[m\to\infty]{} 0,$$

if $i < N_{2m+1}$, and $\|T^i x_{2k}\| < \frac{1}{2m+1}$, $k \leqslant m$. Condition (5) above implies that

$$\frac{1}{N_{2m+1}}\operatorname{card}\left\{0 \leqslant i < N_{2m+1};\ \left\|T^i x_{2k}\right\| < \frac{1}{2m+1},\ k = 1, \dots, m\right\} > 1 - \frac{1}{m},$$

which gives an increasing sequence $A = (n_k)_k$ of integers such that $\overline{\operatorname{dens}}(A) = 1$ and $\lim_k \|T^{n_k}x\| = 0$, concluding the result. $\square$

Observe that Propositions 16 and 20 provide an alternative proof of Theorem 17.

Remark 21. Not every distributionally chaotic operator satisfies the Distributional Chaos Criterion. Indeed, let us consider a weighted forward shift $F_w : \ell^2 \to \ell^2$, defined as $(x_1, x_2, \dots) \mapsto (0, w_1x_1, w_2x_2, \dots)$, where the sequence of weights $w = (w_k)_k$ consists, alternatively, of sufficiently large blocks of 2's and blocks of (1/2)'s such that the vector $e_1 = (1, 0, \dots)$ is a distributionally irregular vector. On the other hand, if $\lim_k F_w^k x = 0$ then $x = 0$.

If we impose that the orbits converge to 0 on a dense subset, then the DCC can be characterized in terms of the existence of distributionally irregular vectors.

Corollary 22. *Let $T : X \to X$ be an operator such that there exists a dense set D with $\lim_{n\to\infty} T^n x = 0$, for all $x \in D$. The following properties are equivalent*:

(i) *T satisfies the Distributional Chaos Criterion.*
(ii) *T admits a distributionally irregular vector.*
(iii) *There exist a sequence $(y_m)_m$ in $X \setminus \{0\}$ and a sequence of positive integers $(N_m)_m$ increasing to ∞, such that*

$$\lim_{m\to\infty}\frac{1}{N_m}\operatorname{card}\left\{0 \leqslant i < N_m;\ \left\|T^i y_m\right\| \geqslant m\|y_m\|\right\} = 1.$$

We can get stronger results under the assumptions above, which depend on the existence of large scrambled sets consisting of (distributionally) irregular vectors.

Definition 23. A linear manifold $Y \subset X$ is a (*distributionally*) *irregular manifold* for $T : X \to X$ if every non-zero vector $y \in Y \setminus \{0\}$ is a (distributionally) irregular vector for T.

Certainly, a (distributionally) irregular manifold is a (distributionally) scrambled set. The extreme case is when the irregular manifold is the whole X. The first construction of an operator for which every non-zero vector is irregular was given by Beauzamy (see pp. 69–70 in [3]), and later it was slightly modified by Prajitura [27]. See also [31] for an example of operator $T : X \to X$ such that the sequence $\{\|T^n x\|\}_n$ is dense in $[0, +\infty[$ for every non-zero $x \in X$.

Theorem 24. *Let $T : X \to X$ be an operator. If there exists a dense subset $X_0 \subset X$ such that $\lim_{n\to\infty} T^n x = 0$, for each $x \in X_0$ and T admits a distributionally irregular vector, then T admits a dense distributionally irregular manifold.*

Proof. We consider a dense sequence $(y_n)_n$ in X_0. By Proposition 20 there is a distributionally irregular vector $x \in X$ for T which can be written as a series

$$x = \sum_{k=1}^{\infty} x_k$$

with the properties specified in the proof. We select a countable collection $(\gamma_m)_m = ((\gamma_{m,k})_k)_m$ of sequences of 0's and 1's such that each sequence γ_m contains an infinite number of 1's and

$$\gamma_{m,k} = 1 \quad \Longrightarrow \quad \gamma_{n,k} = 0, \quad \forall m \neq n,\ \forall k \in \mathbb{N}.$$

We set the sequence of vectors $u_m = \sum_k \gamma_{m,k} x_k$, $m \in \mathbb{N}$, which, by following the proof of Proposition 20, are distributionally irregular vectors since each γ_m contains an infinite number of 1's. Define now

$$z_m = y_m + \frac{1}{m} u_m, \quad m \in \mathbb{N}.$$

Since $(y_n)_n$ is dense in X and the u_m's are uniformly bounded, we get that the sequence $(z_m)_m$ is dense in X. We set $Y = \operatorname{span}\{z_m;\ m \in \mathbb{N}\}$, which is a dense subspace of X. If $y \in Y \setminus \{0\}$, then we can write

$$y = y_0 + \sum_k \rho_k x_k,$$

where $y_0 \in X_0$ and the sequence of scalars $\rho = (\rho_k)_k$ only takes a finite number of values, being non-zero an infinite number of ρ_k's. This means, by following again the proof of Proposition 20, that

$$v := \sum_k \rho_k x_k$$

is a distributionally irregular vector for T. Since $y = y_0 + v$ and $\lim_k T^k y_0 = 0$, we get that y is also a distributionally irregular vector for T, and Y is therefore a distributionally scrambled set for T. □

An adaptation of the above argument, taking into account the proof Theorem 8, yields a useful sufficient condition for dense Li–Yorke chaos.

Theorem 25. *Let $T : X \to X$ be an operator. If $\sup_n \|T^n\| = \infty$ and there is a dense subset $X_0 \subset X$ such that $\lim_{k\to\infty} T^n x = 0$ for each $x \in X_0$, then T admits a dense irregular manifold.*

A class of operators for which Theorems 24 and 25 are particularly interesting and easy to apply is the class of operators with dense generalized kernel.

Corollary 26. *Let $T : X \to X$ be an operator whose generalized kernel $\bigcup_n \ker T^n$ is dense in X.*

(i) *If T has a distributionally irregular vector, then T admits a dense distributionally irregular manifold.*
(ii) *If $\sup_n \|T^n\| = \infty$, then T has a dense irregular manifold.*

To apply Corollary 26 to an important class of operators, we will consider weighted backward shifts $B_w : X \to X$, $(x_0, x_1, \ldots) \mapsto (w_1 x_1, w_2 x_2, \ldots)$ on $X = \ell^p$, $1 \leqslant p < \infty$, or $X = c_0$, where $w = (w_i)_i$ is a bounded sequence of non-zero weights, and the (unweighted) backward shift $B : X \to X$, $(x_0, x_1, \ldots) \mapsto (x_1, x_2, \ldots)$ on the weighted spaces

$$X=\ell^p(v):=\left\{x=(x_i)_i;\ \|x\|:=\left(\sum_i v_i|x_i|^p\right)^{1/p}<\infty\right\},\quad 1\leqslant p<\infty,\quad \text{or}$$

$$X=c_0(v):=\left\{x=(x_i)_i;\ \lim_i v_i x_i=0,\ \|x\|:=\sup_i v_i|x_i|\right\},$$

where $v=(v_i)_i$ is a sequence of strictly positive weights such that $\sup_i v_i/v_{i+1}<\infty$ in order to have a well-defined backward shift bounded operator.

Proposition 27. *The following assertions hold*:

(i) *B is Li–Yorke chaotic if and only if $M_v:=\sup\{v_n/v_m;\ n\in\mathbb{N},\ m>n\}=\infty$. In this case, B admits a dense irregular manifold.*
(ii) *B_w is Li–Yorke chaotic if and only if $M_w:=\sup\{\prod_{k=n}^m|w_k|;\ n\in\mathbb{N},\ m>n\}=\infty$. In this case, B_w admits a dense irregular manifold.*

Proof. Suppose that $B:\ell^p(v)\to\ell^p(v)$ is Li–Yorke chaotic, and let $x=(x_i)_i$ be an irregular vector for B. Since

$$\left\|B^n x\right\|^p=\sum_{k=0}^{\infty}v_k|x_{k+n}|^p=\sum_{k=0}^{\infty}\left(\frac{v_k}{v_{k+n}}\right)v_{k+n}|x_{k+n}|^p\leqslant M_v\|x\|^p,$$

then $M_v<\infty$ implies that $\sup_n\|B^n x\|<\infty$, which is a contradiction. Conversely, if $M_v=\infty$, we find increasing sequences of integers $(n_k)_k$, $(m_k)_k$, $n_k<m_k$, $k\in\mathbb{N}$, such that $\lim_k(m_k-n_k)=\infty$ and $v_{n_k}/v_{m_k}>3^k$, $k\in\mathbb{N}$, by the definition of M_v and the condition on v for continuity of B. We define the vector $x=(x_i)_i\in\ell^p(v)$ by $x_i=(2^k v_{m_k})^{-1/p}$ if $i=m_k$, and $x_i=0$ otherwise. We have

$$\left\|B^{m_k-n_k}x\right\|\geqslant v_{n_k}|x_{m_k}|^p=\left(\frac{v_{n_k}}{v_{m_k}}\right)\frac{1}{2^k}>\left(\frac{3}{2}\right)^k\xrightarrow[k\to\infty]{}+\infty.$$

This shows that B is Li–Yorke chaotic, and it admits a dense linear manifold $Y\subset X$ such that Y is a scrambled set for B by Corollary 26. The case $X=c_0$ is analogous.

(ii) is a consequence of (i) if we proceed by conjugation with a suitable diagonal operator (see, e.g., [21]). □

Another consequence of Theorem 24 is the following easy sufficient condition for dense distributional chaos.

Corollary 28. *If an operator $T:X\to X$ satisfies the following*:

1. *there exist an increasing sequence of integers $B=(m_k)_k$ with $\overline{\mathrm{dens}}(B)=1$, $y\in X$ satisfying $\lim_{k\to\infty}\|T^{m_k}y\|=\infty$, and*
2. *a dense subset $X_0\subset X$ such that $\lim_{n\to\infty}T^n x=0$, for each $x\in X_0$,*

then T admits a dense distributionally irregular manifold.

At this point we want to recall recent result of Müller and Vrsovsky.

Theorem 29. *(See [24].) Let $T_n:X\to X$, $n\in\mathbb{N}$, be a sequence of operators. Suppose that one of the following conditions is satisfied*:

(i) *either $\sum_{n=1}^{\infty}\frac{1}{\|T_n\|}<\infty$;*
(ii) *or X is a complex Hilbert space and $\sum_{n=1}^{\infty}\frac{1}{\|T_n\|^2}<\infty$.*

Then there exists a point $y\in X$ such that $\lim_{n\to\infty}\|T_n y\|=\infty$.

Corollary 28 and Theorem 29 combine nicely to obtain a powerful sufficient condition for dense distributional chaos.

Corollary 30. *Let $T:X\to X$ be an operator such that there exist a dense subset $X_0\subset X$ with $\lim_{n\to\infty}T^n x=0$, for each $x\in X_0$, and an increasing sequence of integers $B=(m_k)_k$ with $\overline{\mathrm{dens}}(B)=1$ satisfying*

(i) *either $\sum_{k=1}^{\infty}\frac{1}{\|T^{m_k}\|}<\infty$;*
(ii) *or X is a complex Hilbert space and $\sum_{k=1}^{\infty}\frac{1}{\|T^{m_k}\|^2}<\infty$.*

Then T has a dense distributionally irregular manifold.

As a consequence of Corollary 30, we obtain the following property.

Corollary 31. *Let $T : X \to X$ be an operator. If there exist a dense set X_0 such that $\lim_{n\to\infty} T^n x = 0$, for each $x \in X_0$ and $r(T) > 1$, then T admits a dense distributionally irregular manifold.*

Definition 32. (See [10].) Let H be a complex separable Hilbert space. For Ω a connected open subset of $\mathbb{C}$ and n a positive integer, let $\mathrm{B}_n(\Omega)$ denote the set of operators T in H which satisfy

(a) $\Omega \subset \sigma(T)$,
(b) $\operatorname{rank}(T - wI) = H$ for $w \in \Omega$,
(c) $\bigvee \ker_{w\in\Omega}(T - wI) = H$,
(d) $\dim\ker(T - wI) = n$ for $w \in \Omega$.

Corollary 33. *Let $T \in \mathrm{B}_n(\Omega)$. If $\Omega \cap \mathbb{T} \neq \emptyset$, then T has a dense distributionally irregular manifold.*

Proof. Let U be a connected open subset of $\Omega \cap \mathbb{D}$. Then $X_0 := \operatorname{span}\bigcup_{w\in U} \ker(T - wI)$ is dense in H and satisfies $\lim_{n\to\infty} \|T^n x\| = 0$, for all $x \in X_0$ and $r(T) > 1$. Hence the conclusion follows by an application of Corollary 31. $\square$

5. Existence of distributionally chaotic operators

The study of general conditions under which a space X admits operators with certain wild behavior has attracted the interest of many researchers. Ansari [1] and Bernal-González [6] independently proved that any separable infinite dimensional Banach space X supports a hypercyclic operator. This result was extended to Fréchet spaces by Bonet and Peris [8]. A generalization for polynomials is given in [23]. Other existence results for related properties and for C_0-semigroups can be found, for instance, in [4,7,5,29,30].

In this section we want to establish the existence of distributionally chaotic operators on arbitrary infinite dimensional and separable Banach spaces. The following lemma of Shkarin [29] and its consequences will be a key to obtain our existence result.

Lemma 34. *(See [29].) Let X be a separable infinite dimensional Banach space, let $(b_k)_k$ be a sequence of numbers in $[3,+\infty[$ such that $b_k \to \infty$ as $k \to \infty$ and $(N_k)_k$ be a strictly increasing sequence of positive integers such that $N_0 = 0$ and $N_{k+1} - N_k \geqslant 2$ for each $k \in \mathbb{N}$. Then there exists a biorthogonal sequence $((y_k, f_k))_k$ in $X \times X^*$ such that*

(B1) $\|y_k\| = 1$ *for each* $k \in \mathbb{N}$;
(B2) $\operatorname{span}\{y_k\colon k \in \mathbb{N}\}$ *is dense in* X;
(B3) $\|f_{N_k}\| \leqslant b_k$ *for each* $k \in \mathbb{N}$;
(B4) $\|f_j\| \leqslant 3$ *if* $j \in \mathbb{N} \setminus \{N_k\colon k \in \mathbb{N}\}$;
(B5) *for any* $k \in \mathbb{Z}$ *and any* $c_j \in \mathbb{K}$ *with* $N_k + 1 \leqslant j \leqslant N_{k+1} - 1$

$$\frac{1}{2}\left\| \sum_{j=N_k+1}^{N_{k+1}-1} c_j y_j \right\| \leqslant \left(\sum_{j=N_k+1}^{N_{k+1}-1} |c_j|^2 \right)^{\frac{1}{2}} \leqslant 2\left\| \sum_{j=N_k+1}^{N_{k+1}-1} c_j y_j \right\|.$$

In Section 3 we showed that there are no operators satisfying the Strong Distributional Chaos Criterion on certain Banach spaces. In the following result we give a positive answer for distributionally chaotic operators, obtaining the extra property of hypercyclicity. Separability of the underlying space is, therefore, necessary for the existence of dense orbits.

Theorem 35. *In every infinite dimensional separable Banach space there exists a hypercyclic and distributionally chaotic operator which admits a dense distributionally irregular manifold.*

Proof. By [29, Lemma 2.5], given a sequence $(w_n)_n$ of positive weights in ℓ^2 there exists $T : X \to X$ satisfying $Ty_0 = 0$ and $Ty_{n+1} = w_n y_n$ with $\|y_n\| = 1$ where $(y_n)_n$ is given by Lemma 34.

We need to compute $\|(I+T)^i y_j\|$, that is

$$\begin{aligned}
\|(I+T)^i y_j\| &= \left\| \sum_{k=0}^{i} \binom{i}{k} T^k y_j \right\| \\
&= \left\| y_j + \binom{i}{1} T y_j + \cdots + \binom{i}{i} T^i y_j \right\| \\
&= \|y_j + i w_{j-1} y_{j-1} + \cdots + w_{j-1} \cdots w_{j-i} y_{j-i}\|.
\end{aligned}$$

If $j = N_{k+1} - 1$ and $j - i \geqslant N_k + 1$, that is, $i \leqslant N_{k+1} - N_k - 2$, then

$$\left\|(I+T)^i y_j\right\| \geqslant \frac{1}{2} i w_{j-1}$$

by condition (B5) in Lemma 34. Taking $w_k = k^{-\frac{2}{3}}$, $N_0 := 0$ and $N_k = (k+1)! + 1$ for $k \geqslant 1$, then for $i = 3k([((k+2)!)^{\frac{2}{3}}] + 1), \dots, (k+2)! - (k+1)! - 2$

$$\left\|(I+T)^i y_j\right\| \geqslant \frac{1}{2} i w_{j-1} \geqslant \frac{1}{2} \frac{3k([((k+2)!)^{\frac{2}{3}}] + 1)}{((k+2)! - 1)^{\frac{2}{3}}} > k+1,$$

for $j = (k+2)! = N_{k+1} - 1$. Thus, for $u_m = y_{N_{m+1}-1}$, $m \in \mathbb{N}$, we have

$$\begin{aligned} &\frac{1}{N_m} \operatorname{card}\left\{0 \leqslant i < N_m \colon \left\|(I+T)^i u_m\right\| \geqslant m \|u_m\| = m\right\} \\ &\quad \geqslant \frac{(m+1)! - m! - 2 - (3(m-1)([((m+1)!)^{\frac{2}{3}}] + 1))}{(m+1)! + 1} \xrightarrow[m \to \infty]{} 1. \end{aligned}$$

Moreover, $I+T$ satisfies the Kitai Criterion (therefore, it is hypercyclic) [29, Theorem 1.1 and Corollary 1.3], thus there exists a dense sequence $(x_k)_k$ in X such that $(I+T)^n x_k \to 0$ as $n \to \infty$.

Hence the operator $I+T$ satisfies the conditions of Corollary 22 and Theorem 24, and we conclude that $I+T$ has a dense distributionally irregular manifold. □

Acknowledgments

The second author is supported in part by MEC and FEDER, Project MTM2008-05891. The third and fourth authors are supported in part by MEC and FEDER, Projects MTM2007-64222, MTM2010-14909 and by Generalitat Velenciana, Project PROMETEO/2008/101. We want to thank the referee for the suggestions.

References

[1] S. Ansari, Existence of hypercyclic operators on topological vector spaces, J. Funct. Anal. 148 (1997) 384–390.
[2] F. Bayart, É. Matheron, Dynamics of Linear Operators, vol. 179, Cambridge University Press, 2009.
[3] B. Beauzamy, Introduction to Operator Theory and Invariant Subspaces, North-Holland, Amsterdam, 1988.
[4] T. Bermúdez, A. Bonilla, A. Martinón, On the existence of chaotic and hypercyclic semigroups on Banach spaces, Proc. Amer. Math. Soc. 131 (2003) 2435–2441.
[5] T. Bermúdez, A. Bonilla, J.A. Conejero, A. Peris, Hypercyclic, topologically mixing and chaotic semigroups on Banach spaces, Studia Math. 170 (2005) 57–75.
[6] L. Bernal-González, On hypercyclic operators on Banach spaces, Proc. Amer. Math. Soc. 127 (1999) 1003–1010.
[7] L. Bernal-González, K.-G. Grosse-Erdmann, Existence and nonexistence of hypercyclic semigroups, Proc. Amer. Math. Soc. 135 (2007) 755–766.
[8] J. Bonet, A. Peris, Hypercyclic operators on non-normable Fréchet spaces, J. Funct. Anal. 159 (1998) 587–595.
[9] J. Bonet, F. Martínez-Giménez, A. Peris, A Banach space which admits no chaotic operator, Bull. London Math. Soc. 33 (2001) 196–198.
[10] M.J. Cowen, R.G. Douglas, Complex geometry and operator theory, Acta Math. 141 (1) (1978) 187–261.
[11] R.L. Devaney, An Introduction to Chaotic Dynamical Systems, 2nd edition, Addison–Wesley Stud. Nonlinearity, Addison–Wesley, Redwood City, CA, 1989.
[12] J. Duan, X.-C. Fu, P.-D. Liu, A. Manning, A linear chaotic quantum harmonic oscillator, Appl. Math. Lett. 12 (1999) 15–19.
[13] N.S. Feldman, T.L. Miller, V.G. Miller, Hypercyclic and supercyclic cohyponormal operators, Acta Sci. Math. (Szeged) 68 (2002) 303–328.
[14] X.-C. Fu, J. Duan, Infinite-dimensional linear dynamical systems with chaoticity, J. Nonlinear Sci. 9 (1999) 197–211.
[15] W.T. Gowers, B. Maurey, The unconditional basic sequence problem, J. Amer. Math. Soc. 6 (1993) 851–874.
[16] K.G. Grosse-Erdmann, A. Peris, Linear Chaos, Universitext, Springer, in press.
[17] B. Hou, P. Cui, Y. Cao, Chaos for Cowen–Douglas operators, Proc. Amer. Math. Soc. 138 (2010) 929–936.
[18] B. Hou, G. Tian, L. Shi, Some dynamical properties for linear operators, preprint, arXiv:0903.4558v1.
[19] B. Hou, L. Shi, G. Tian, S. Zuh, Approximation of chaotic operators, J. Operator Theory, in press.
[20] T.Y. Li, J.A. Yorke, Period three implies chaos, Amer. Math. Monthly 82 (1975) 985–992.
[21] F. Martínez-Giménez, A. Peris, Chaos for backward shift operators, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 12 (2002) 1703–1715.
[22] F. Martínez-Giménez, P. Oprocha, A. Peris, Distributional chaos for backward shifts, J. Math. Anal. Appl. 351 (2009) 607–615.
[23] F. Martínez-Giménez, A. Peris, Existence of hypercyclic polynomials on complex Fréchet spaces, Topology Appl. 156 (2009) 3007–3010.
[24] V. Müller, J. Vrsovsky, Orbits of linear operators tending to infinity, Rocky Mountain J. Math. 39 (2009) 219–230.
[25] P. Oprocha, A quantum harmonic oscillator and strong chaos, J. Phys. A 39 (2006) 14559–14565.
[26] P. Oprocha, Distributional chaos revisited, Trans. Amer. Math. Soc. 361 (2009) 4901–4925.
[27] G.T. Prăjitură, Irregular vectors of Hilbert space operators, J. Math. Anal. Appl. 354 (2009) 689–697.
[28] B. Schweizer, J. Smítal, Measures of chaos and a spectral decomposition of dynamical systems on the interval, Trans. Amer. Math. Soc. 344 (1994) 737–754.
[29] S. Shkarin, The Kitai criterion and backward shifts, Proc. Amer. Math. Soc. 136 (2008) 1659–1670.
[30] S. Shkarin, A short proof of existence of disjoint hypercyclic operators, J. Math. Anal. Appl. 367 (2010) 713–715.
[31] L. Smith, A nonhypercyclic operator with orbit-density properties, Acta Sci. Math. (Szeged) 74 (2008) 743–756.